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Error Bounds for Spline Interpolation over Rectangular Polygons

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1. INTRODUCTION

This paper is concerned with the determination of error bounds for bicubic and biquintic spline interpolation over rectangular polygons. We show that for smooth functions, the rate of convergence of the bicubic and biquintic spline interpolants constructed in [12] on simple rectangular polygons (explicitly: L-shaped and U-shaped regions) is of fourth order in the bicubic case and of sixth order in the biquintic case, independently of mesh ratios. This agrees with the results of Carlson and Hall [7] concerning bicubic spline interpolation on a rectangle. For functions $f \in C^4$, they prove the fourth-order convergence of bicubic spline interpolants over rectangles, independently of mesh ratios. In addition, they obtain explicit bounds on the constants involved. If one does not insist on such explicit bounds, one can easily obtain analogous results in the biquintic case using tensor products and the results of de Boor [3, 4].

The results of Carlson and Hall [7] in turn extend the univariate result that for $f \in C^{(2p)}[a, b]$, $p = 2, 3$, one obtains the error bound of best possible order

$$\|(f - I_\pi^p f)^{(j)}\|_\infty \leqslant K\bar{\pi}^{2p-j} \|f^{(2p)}\|_\infty, \qquad 0 \leqslant j \leqslant p, \tag{1.1}$$

where $I_\pi^p f$ is the spline interpolant of degree $2p - 1$ to f satisfying

$$(I_\pi^p f)(x_i) = f(x_i), \qquad i = 0,..., M, \tag{1.2}$$

$$(I_\pi^p f)^{(j)}(x_i) = f^{(j)}(x_i), \qquad i = 0, M; \quad j = 1,..., p - 1, \tag{1.3}$$

where $\bar{\pi} = \max_i | x_{i+1} - x_i |$, and K is a constant which is independent of f and π. For $p = 2$, this was proved in [2], [4], [9], and for $p = 3$ in [4].

In this paper, we let $\mathscr{R}$ be a rectangular polygon and $R = [a, b] \times [c, d]$ be the smallest rectangle containing $\mathscr{R}$. Let $\pi = \pi_1 \times \pi_2$ be a rectangular mesh defined on R containing all corners of $\mathscr{R}$ and R as mesh points. We also set $x_0 = a$, $x_M = b$, $y_0 = c$, $y_N = d$. Let $S^{2p}(\mathscr{R}, \pi)$ denote the subspace of piecewise polynomials, or splines, on π (or more precisely the restriction

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of π to $\mathscr{R}$) which are of degree $2p-1$ in each variable and which belong to $C^{(2p-2,2p-2)}(\mathscr{R})$, the class of functions f defined on $\mathscr{R}$ with $f^{(i,j)}$ continuous on $\mathscr{R}$, $0 \leqslant i, j \leqslant 2p-2$. In [12] a linear interpolation projector $I^{p}_{\pi,\mathscr{R}}$ onto $S^{2p}(\mathscr{R}, \pi)$ was constructed having the property that the spline interpolant determined by this projector can be characterized by variational properties. This generalizes the well-known result concerning spline interpolation in one variable [1], and its extension by tensor products to rectangles [11, 12].

Except along edges and corners of $\mathscr{R}$ which do not coincide with the rectangle R, the interpolation conditions which define the interpolant of [12] are identical to the conditions defining the spline interpolant of the same degree on R obtained from univariate spline interpolation by taking tensor products. Along these edges and corners, simple interpolation to normal and cross-derivatives must be replaced by certain linear combinations of interpolation to values and derivatives in order to obtain an interpolant in $S^{2p}(\mathscr{R}, \pi)$ which can be characterized by variational properties. A precise description of these interpolation conditions is given in Section 2. Also in Section 2, we extend to general rectangular polygons a result for L-shaped regions based on the variational property obtained in [11, Theorem 5]. We do this not only for its own interest but because most of the estimates used to prove this result will be needed to prove our main results in Sections 3 and 4.

2. Convergence Results Based on the Variational Property

On $C^{(p,p)}(R)$ we define the inner product

$$(f, g) = [f, g] + \sum_{i=1}^{p} \sum_{j=1}^{p} [f(\bar{x}_i, \bar{y}_j)][g(\bar{x}_i, \bar{y}_j)], \tag{2.1}$$

where

$$[f, g] = \int_a^b \int_c^d f^{(p,p)}(x, y)\, g^{(p,p)}(x, y)\, dx\, dy + \sum_{j=1}^{p} \int_a^b f^{(p,0)}(x, \bar{y}_j)\, g^{(p,0)}(x, \bar{y}_j)$$
$$+ \sum_{i=1}^{p} \int_c^d f^{(0,p)}(\bar{x}_i, y)\, g^{(0,p)}(\bar{x}_i, y)\, dy, \tag{2.2}$$

where the $\bar{x}_i$ and $\bar{y}_j$ are chosen so that $(\bar{x}_i, \bar{y}_j)$, $1 \leqslant i, j \leqslant p$, are contained in $\mathscr{R}$. The completion of $C^{(p,p)}(R)$ with respect to the inner product (2.1) is the Hilbert space $R_C^{p,p}$ with the properties

$f^{(i,j)} \in C[R], \quad i < p, \quad j < p,$
$f^{(p-1,0)}(x, \bar{y}_j)$ is abs. cont., $f^{(p,0)}(x, \bar{y}_j) \in L^2[a, b], \qquad j = 1,\ldots, p,$
$f^{(0,p-1)}(\bar{x}_i, y)$ is abs. cont., $f^{(0,p)}(\bar{x}_i, y) \in L^2[c, d], \qquad i = 1,\ldots, p,$
$f^{(p-1,p-1)}$ is abs. cont., $f^{(p,p)} \in L^2[R]$.

Let E be the linear map on $C^{(p,p)}(\mathscr{R})$ given by

$$Ef = \begin{cases} f & \text{on } \mathscr{R} \\ Vf & \text{on } R\backslash\mathscr{R}, \end{cases} \tag{2.3}$$

where V is a linear map satisfying

$$\frac{\partial^i(Vf)}{\partial n^i} = \frac{\partial^i f}{\partial n^i}, \qquad 0 \leqslant i \leqslant p-1,$$

on $\partial\mathscr{R} \cap \partial(R\backslash\mathscr{R})$, where $\partial/\partial n$ denotes the normal derivative. Thus $E[C^{(p,p)}(\mathscr{R})] \subset R_C^{p,p}$. On $C^{(p,p)}(\mathscr{R})$ we define the inner product

$$(f, g)_* = (Ef, Eg). \tag{2.4}$$

The completion of $C^{(p,p)}(\mathscr{R})$ with respect to the inner product (2.4) is the Hilbert space $\mathscr{R}^{p,p}$. The linear operator E then can of course be extended by continuity to an operator defined on $\mathscr{R}^{p,p}$.

DEFINITION. We say that E defines a *minimal extension* of $\mathscr{R}^{p,p}$ to $R_C^{p,p}$ if its left inverse F defined by

$$Ff = F\,|_{\mathscr{R}}$$

is norm reducing.

We now define the spline interpolant $I^p_{\pi,\mathscr{R}}f$. The functionals, or interpolation conditions, which determine the projector $I^p_{\pi,\mathscr{R}}$ are defined in terms of the minimal extension E and the functionals which determine the projector $I^p_{\pi,R} = I^p_{\pi_1} \otimes I^p_{\pi_2}$ with range $S^{2p}(R, \pi)$. Here $I^p_{\pi_1}$ is the projector defined by (1.2)–(1.3). Let the functionals $\delta^{(j)}_\alpha$ be defined by

$$\delta^{(j)}_\alpha g = g^{(j)}(\alpha), \qquad j < p,$$

with $\delta_\alpha = \delta^{(0)}_\alpha$.

The functionals which determine the spline interpolant $I^p_{\pi,R}f$ are

$$\Lambda = \{\lambda \otimes \mu \mid \lambda \in \Lambda_1, \mu \in \Lambda_2\},$$

where

$$\Lambda_1 = \{\delta_{x_i}\}_0^M \cup \{\delta^{(i)}_a\}_1^{p-1} \cup \{\delta^{(i)}_b\}_1^{p-1},$$

and

$$\Lambda_2 = \{\delta_{y_j}\}_0^N \cup \{\delta^{(j)}_c\}_1^{p-1} \cup \{\delta^{(j)}_d\}_1^{p-1}.$$

Let $m = \dim S^{2p}(\mathscr{R}, \pi)$ and assume that the mesh π is suitably refined so that $\dim S^{2p}(\mathscr{R}, \pi) \leqslant \dim S^{2p}(R, \pi)$. See [6, Section 3] for a discussion of this restriction. The spline interpolant $I^p_{\pi,\mathscr{R}}f$ to $f \in \mathscr{R}^{p,p}$ satisfies the interpolation conditions

$$\mu_i I^p_{\pi,\mathscr{R}} f = \mu_i f, \qquad i = 1,..., m, \tag{2.5}$$

where $\{\mu_i\}_1^m$ is any set of m linearly independent functionals from the set $\{\lambda E \mid \lambda \in \Lambda\}$. The set $\{\mu_i\}_1^m$ can always be chosen to include interpolation to values at each mesh point of π which is contained in $\mathscr{R}$ plus certain linear combinations of these functionals and certain functionals associated with interpolation to derivatives on the boundary of $\mathscr{R}$. For the case of the L-shaped region these interpolation conditions are explicitly given in [12, Theorem 2]. For the U-shaped region they are explicitly given in Section 4. It is shown in [12, Section 6] that the linear projection $I^p_{\pi,\mathscr{R}}f$ of $f \in \mathscr{R}^{p,p}$ onto $S^{2p}(\mathscr{R}, \pi)$ satisfying (2.5) uniquely minimizes $[v, v]_* = [Ev, Ev]$ among all $v \in \mathscr{R}^{p,p}$ satisfying

$$\mu_i v = \mu_i f, \qquad i = i,..., m.$$

We are now ready to state and prove the main result of this section. Let $\bar{\pi}_x = \max_i | x_{i+1} - x_i |$, $\bar{\pi}_y = \max_j | y_{j+1} - y_j |$ and $\bar{\pi} = \max\{\bar{\pi}_x, \bar{\pi}_y\}$. In addition let

$$\| g \|_{\infty,\mathscr{R}} = \sup\{| g(t, u)|: (t, u) \in \mathscr{R}\}.$$

THEOREM 1. *Let $f \in \mathscr{R}^{p,p}$ and $r = f - I^p_{\pi,\mathscr{R}}f$. Then there exists a constant K independent of f and π such that*

$$\| r^{(i,j)} \|_{\infty,\mathscr{R}} \leqslant K\bar{\pi}^{p-(1/2)-k}[Ef, Ef]^{1/2}, \qquad k = \max\{i, j\}, \qquad 0 \leqslant i, \; j \leqslant p - 1. \tag{2.6}$$

Remark. The theorem will be proved by showing that for each $(x, y) \in \mathscr{R}$, $[(1 - Q_{(x,y)})r]^{(i,j)}$ can be bounded, using a modification of the Sard kernel theorem, independently of x and y in terms of $\bar{\pi}^{p-(1/2)-k}$, where $Q_{(x,y)}$ is a projector defined on R with $Q_{(x,y)}r \equiv 0$. All of the bounds established in the proof except for the final inequalities will be needed to establish our results in Sections 3 and 4 concerning spline interpolation on L-shaped and U-shaped regions.

Proof of Theorem 1. Let P_1 be the linear projector of Lagrange interpolation onto $\mathscr{P}_{p-1}$, the space of polynomials of degree $p - 1$ or less, defined by

$$(P_1 g)(x) = \sum_{i=1}^{p} l_i(x)\, g(\bar{x}_i), \tag{2.7}$$

where

$$l_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{p} (x - \bar{x}_j)/(\bar{x}_i - \bar{x}_j), \qquad 1 \leqslant i \leqslant p. \tag{2.8}$$

Likewise, let P_2 be the linear projector onto $\mathscr{P}_{p-1}$ defined by

$$(P_2 g)(y) = \sum_{j=1}^{p} \tilde{l}_j(y)\, g(\bar{y}_j), \tag{2.9}$$

where

$$\tilde{l}_j(y) = \prod_{\substack{i=1\\ i\neq j}}^{p} (y - \bar{y}_i)/(\bar{y}_j - \bar{y}_i), \qquad 1 \leqslant j \leqslant p. \tag{2.10}$$

Let (x, y) be an arbitrary point in $\mathscr{R}$. Then by the Peano kernel theorem,

$$\begin{aligned} r(x, y) = (P_1 \otimes P_2)(Er)(x, y) &+ \sum_{j=1}^{p} \tilde{l}_j(y) \int_a^b R(x, t)(Er)^{(p,0)}(t, \bar{y}_j)\, dt \\ &+ \sum_{i=1}^{p} l_i(x) \int_c^d \tilde{R}(y, u)(Er)^{(0,p)}(\bar{x}_i, u)\, du \\ &+ \int_a^b \int_c^d R(x, t)\, \tilde{R}(y, u)(Er)^{(p,p)}(t, u)\, dt\, du \end{aligned} \tag{2.11}$$

where

$$R(x, t) = (1 - P_1)_{(x)}\, g(x, t),$$
$$g(x, t) = (x - t)_+^{p-1}/(p - 1)!,$$

and

$$\tilde{R}(y, u) = (1 - P_2)_{(y)}\, \tilde{g}(y, u),$$
$$\tilde{g}(y, u) = (y - u)_+^{p-1}/(p - 1)!.$$

There exists $(x_\nu, y_\mu) \in \pi$ such that $(x, y) \in [x_\nu, x_{\nu+p-1}] \times [y_\mu, y_{\mu+p-1}] \subset \mathscr{R}$. Let

$$Q = Q_{(x,y)} = Q_1 \otimes Q_2,$$

where Q_1 is the projector defined by

$$Q_1 v(x) = \sum_{i=0}^{p-1} c_i(x)\, v(x_{\nu+i}),$$

where

$$c_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{p-1} (x - x_{\nu+j})/(x_{\nu+i} - x_{\nu+j}), \qquad i = 0,..., p - 1,$$

and Q_2 is the projector defined by

$$Q_2 v(y) = \sum_{j=0}^{p-1} \tilde{c}_j(y)\, v(y_{\mu+j}),$$

where

$$\tilde{c}_j(y) = \prod_{\substack{i=0\\ i\neq j}}^{p-1} (y - y_{\mu+i})/(y_{\mu+j} - y_{\mu+i}), \qquad j = 0,\dots, p-1.$$

Then using Lemma 2 of [11] which is a minor modification of the Sard kernel theorem [13, p. 200], $r^{(i,j)}(x, y)$ can be expressed as

$$\begin{aligned} r^{(i,j)}(x, y) = {} & \sum_{n=1}^{p} \int_a^b \partial^{i+j}/\partial x^i\, \partial y^j (K_{p,n-1}(x, y, t))(Er)^{(p,0)}(t, \bar{y}_n)\, dt \\ & + \sum_{k=1}^{p} \int_c^d \partial^{i+j}/\partial x^i\, \partial y^j (K_{k-1,p}(x, y, u))(Er)^{(0,p)}(\bar{x}_k, u)\, du \\ & + \int_a^b \int_c^d \partial^{i+j}/\partial x^i\, \partial y^j (K_{p,p}(x, y, t, u))(Er)^{(p,p)}(t, u)\, dt\, du, \\ & \qquad 0 \leqslant i,\ j \leqslant p-1, \end{aligned} \tag{2.12}$$

where

$$K_{p,n-1}(x, y, t) = (1 - Q)_{(x,y)}\, (R(x, t)\, \tilde{l}_n(y)), \qquad n = 1,\dots, p, \tag{2.13}$$

$$K_{k-1,p}(x, y, u) = (1 - Q)_{(x,y)}\, (l_k(x)\, \tilde{R}(y, u)), \qquad k = 1,\dots, p, \tag{2.14}$$

and

$$K_{p,p}(x, y, t, u) = (1 - Q)_{(x,y)}\, (R(x, t)\, \tilde{R}(y, u)). \tag{2.15}$$

We now state as lemma results proved in [11] which serve to bound the kernels $K_{k,n}$ and their first $p - 1$ derivatives in each variable.

LEMMA 1. *There exist constants M_s independent of $(x, y) \in \mathscr{R}$ such that for $0 \leqslant i, j \leqslant p - 1$, $(t, u) \in R$*

(i) $|\, \partial^{i+j}/\partial x^i\, \partial y^j\, K_{p,n-1}(x, y, t)| \leqslant M_1 \bar{\pi}^{p-1-i}$, $n = 1,\dots, p$,

(ii) $|\, \partial^{i+j}/\partial x^i\, \partial y^j\, K_{k-1,p}(x, y, u)| \leqslant M_2 \bar{\pi}^{p-1-j}$, $k = 1,\dots, p$,

(iii) $|\, \partial^{i+j}/\partial x^i\, \partial y^j\, K_{p,p}(x, y, t, u)| \leqslant M_3 \bar{\pi}^{p-1-l}$, $l = \max\{i, j\}$.

In addition for fixed $(x, y) \in \mathscr{R}$, $K_{p,n-1}(x, y, t)$ is nonzero only on the interval $x_\nu \leqslant t \leqslant x_{\nu+p-1}$, $1 \leqslant n \leqslant p$; $K_{k-1,p}(x, y, u)$ is nonzero only on the interval $y_\mu \leqslant u \leqslant y_{\mu+p-1}$, $1 \leqslant k \leqslant p$; and $K_{p,p}(x, y, t, u)$ is nonzero only for values $(t, u) \in [x_\nu, x_{\nu+p-1}] \times [y_\mu, y_{\mu+p-1}]$.

By Lemma 1 of [11], $I^p_{\pi,\mathscr{R}} f$ is the orthogonal projection of f onto $S^{2p}(\mathscr{R}, \pi)$ with respect to the norm obtained from the inner product (2.4). Thus $[Er, Er] \leqslant [Ef, Ef]$. This, along with the Schwartz inequality and Lemma 1 above applied to (2.12) gives (2.6) and proves the theorem.

3. Convergence of Spline Interpolation over L-Shaped Regions

Let $\mathscr{L}$ be the L-shaped region with reentrant corner (α, β) and other corners at (a, c), (b, c), (b, β), (a, d), (α, d).

Let E be defined by

$$Ef = \begin{cases} f & \text{on } \mathscr{L} \\ Tf & \text{on } R\backslash\mathscr{L}, \end{cases} \tag{3.1}$$

where

$$T = (T_{\alpha,p} \otimes 1) + (1 \otimes T_{\beta,p}) - (T_{\alpha,p} \otimes T_{\beta,p}),$$

with

$$(T_{\alpha,p} g)(x) = \sum_{j<p} g^{(j)}(\alpha) \frac{(x-\alpha)^j}{j!}.$$

Since

$$\frac{\partial^i(Tf)}{\partial n^i} = \frac{\partial^i f}{\partial n^i}, \qquad 0 \leqslant i \leqslant p-1,$$

on $\partial\mathscr{L} \cap \partial(R\backslash\mathscr{L})$ and $(Tf)^{(p,p)} \equiv 0$, it is clear that (3.1) defines a minimal extension for $\mathscr{L}$.

Thus

$$[Ef, Ef] = \iint_{\mathscr{L}} [f^{(p,p)}(x,y)]^2\, dx\, dy + \sum_{j=1}^{p} \int_a^b [f^{(p,0)}(x, \bar{y}_j)]^2\, dx$$

$$+ \sum_{i=1}^{p} \int_c^d [f^{(0,p)}(\bar{x}_i, y)]^2\, dy \tag{3.2}$$

where the $\bar{x}_i$ are chosen from the interval $[a, \alpha]$ and the $\bar{y}_j$ are chosen from $[c, \beta]$.

The main result of the previous section was obtained by applying the Schwartz inequality to the right hand side of (2.12). In this section we apply appropriate Hölder inequalities to each term on the right hand side of (2.12) individually. Thus by Lemma 1

$$\left\| \int_a^b \partial^{i+j}/\partial x^i\, \partial y^j (K_{p,n-1}(x,y,t))(Er)^{(p,0)}(t, \bar{y}_n)\, dt \right\|_{\infty,\mathscr{L}}$$

$$\leqslant (p-1)\, M_1 \bar{\pi}^{p-i} \| r^{(p,0)}(t, \bar{y}_n)\|_\infty, \qquad n = 1,\ldots, p, \tag{3.3}$$

$$\left\| \int_c^d \partial^{i+j}/\partial x^i\, \partial y^j (K_{k-1,p}(x,y,u))(Er)^{(0,p)}(\bar{x}_k, u)\, du \right\|_{\infty,\mathscr{L}}$$

$$\leqslant (p-1)\, M_2 \bar{\pi}^{p-j} \| r^{(0,p)}(\bar{x}_k, u)\|_\infty, \qquad k = 1,\ldots, p, \tag{3.4}$$

and

$$\left\| \int_a^b \int_c^d \partial^{i+j}/\partial x^i\, \partial y^j (K_{p,p}(x, y, t, u))(Er)^{(p,p)}(t, u)\, dt\, du \right\|_{\infty,\mathscr{L}}$$
$$\leqslant (p-1)\, M_3 \bar{\pi}^{p-l} \| r^{(p,p)} \|_{L^2(\mathscr{L})}, \qquad l = \max\{i, j\}. \tag{3.5}$$

THEOREM 2. *Let $f^{(p,p)} \in C^{(p)}(\mathscr{L})$, $p = 2, 3$. Then there exist constants K_μ independent of π and f such that*

$$\| r^{(i,j)} \|_{\infty,\mathscr{L}} \leqslant K_1 \bar{\pi}^{2p-i} \| f^{(2p,0)} \|_{\infty,\mathscr{L}} + K_2 \bar{\pi}^{2p-j} \| f^{(0,2p)} \|_{\infty,\mathscr{L}}$$
$$+ K_3 \bar{\pi}^{2p-l} \max_{|s|=p} \| D^s (f^{(p,p)}(x, y)) \|_{\infty,\mathscr{L}},$$
$$l = \max\{i, j\}, \quad 0 \leqslant i, \quad j \leqslant p-1, \tag{3.6}$$

where $r = f - I^p_{\pi,\mathscr{L}} f$.

Proof. The proof consists of bounding the quantities $\| r^{(p,0)}(t, \bar{y}_n) \|_\infty$, $\| r^{(0,p)}(\bar{x}_k, u) \|_\infty$, and $\| r^{(p,p)} \|_{L^2(\mathscr{L})}$ in terms of $\bar{\pi}$. First, on the line $y = \bar{y}_n$, $I^p_{\pi,\mathscr{L}} f$ is identical to $I^p_{\pi_1} f(x, \bar{y}_n)$, the univariate spline interpolant to $f(x, \bar{y}_n)$ with respect to π_1. The results of de Boor [4] imply that for $p = 2, 3$, $g \in C^{(2p)}[a, b]$,

$$\| (g - I^p_{\pi_1} g)^{(p)} \|_\infty \leqslant K_p \bar{\pi}_1{}^p \| g^{(2p)} \|_\infty, \tag{3.7}$$

where K_p is independent of g and π_1. The first two terms of the right side of (3.6) follow immediately from the application of (3.7) to (3.3) and (3.4) along with summation over n and k.

LEMMA 2. *Let $s_f = I^p_{\pi,\mathscr{L}} f$. For all $s \in S^{2p}(\mathscr{L}, \pi)$*

$$\| r^{(p,p)} \|_{L^2(\mathscr{L})} \leqslant \| f^{(p,p)} - s^{(p,p)} \|_{L^2(\mathscr{L})} \tag{3.8}$$

with equality if and only if $s = s_f + \eta$ where $\eta^{(p,p)} \equiv 0$ on $\mathscr{L}$.

Proof. Let $v \in S^{2p}(\mathscr{L}, \pi)$ and let

$$\tilde{v} = v - \sum_{i=1}^{p} l_i(x)[v(\bar{x}_i, y) - s_f(\bar{x}_i, y)] - \sum_{j=1}^{p} \tilde{l}_j(y)[v(x, \bar{y}_j) - s_f(x, \bar{y}_j)]$$
$$+ \sum_{i=1}^{p} \sum_{j=1}^{p} l_i(x)\, \tilde{l}_j(y)[v(\bar{x}_i, \bar{y}_j) - s_f(\bar{x}_i, \bar{y}_j)],$$

where the $l_i(x)$ and $\tilde{l}_j(y)$ are defined by (2.8) and (2.10). Thus, $\tilde{v}$ agrees with s_f along the lines $y = \bar{y}_j$ and $x = \bar{x}_i$. Also $\tilde{v}^{(p,p)} \equiv v^{(p,p)}$. Therefore

$$\| f^{(p,p)} - v^{(p,p)} \|_{L^2(\mathscr{L})} - \| r^{(p,p)} \|_{L^2(\mathscr{L})} = (f - \tilde{v}, f - \tilde{v})_* - (r, r)_*,$$

which is nonnegative with equality if and only if $\tilde{v} = s_f$, since by Lemma 1 of [12] and the discussion in Section 4 of [12], s_f is the orthogonal projection of f onto $S^{2p}(\mathscr{L}, \pi)$ with respect to the norm $\| g \|^2 = (g, g)_*$. Q.E.D.

Lemma 2 implies that

$$(I^p_{\pi,\mathscr{L}} f)^{(p,p)} = L^p_{\pi,\mathscr{L}} f^{(p,p)},$$

where $L^p_{\pi,\mathscr{L}}$ is the linear projector on $L^2(\mathscr{L})$ which associates with each $g \in L^2(\mathscr{L})$ its best approximation $L^p_{\pi,\mathscr{L}} g$ in $S^p(\mathscr{L}, \pi)$ with respect to the norm $\| g \|_{L^2(\mathscr{L})}$. Rather than attempt to bound

$$\| f^{(p,p)} - L^p_{\pi,\mathscr{L}} f^{(p,p)} \|_{L^2(\mathscr{L})} \tag{3.9}$$

directly in terms of $\bar{\pi}^p$, we instead shall find it easier to bound

$$\| f^{(p,p)} - \hat{s} \|_{L^2(\mathscr{L})}, \tag{3.10}$$

where $\hat{s}$ is the quasi-interpolant of degree $p - 1$ to $f^{(p,p)}$ as defined in [5]. The results of [5] imply that

$$\| f^{(p,p)} - \hat{s} \|_{\infty,\mathscr{L}} \leqslant K_3' \bar{\pi}^p \max_{|\mu|=p} \| D^\mu (f^{(p,p)}(x, y)) \|_{\infty,\mathscr{L}}, \tag{3.11}$$

where K_3' is some constant independent of $\bar{\pi}$ and $f^{(p,p)}$. The last bound in (3.6) now immediately follows with $K_3 = (p - 1) M_3 K_3' (\text{mes}(\mathscr{L}))^{1/2}$. Q.E.D.

4. Convergence of Spline Interpolation over U-Shaped Regions

The proof of Theorem 2 was facilitated by the fact that $\|(Ef)^{(p,p)}\|_{L^2(R\setminus\mathscr{L})}$ was zero. We next consider a U-shaped region, the simplest rectangular polygon where this simplification does not occur.

Let $\mathscr{U}$ be the U-shaped region shown in Figure 1

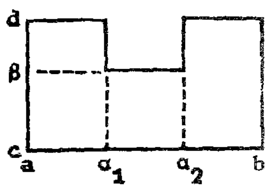

Figure 1

Let E be defined by

$$Ef = \begin{cases} f & \text{on } \mathscr{U} \\ Hf & \text{on } R\setminus\mathscr{U}, \end{cases} \tag{4.1}$$

where

$$H = (H_{\alpha_1,\alpha_2,p} \otimes 1) + (1 \otimes T_{\beta,p}) - (H_{\alpha_1,\alpha_2,p} \otimes T_{\beta,p}),$$

where $H_{\alpha_1,\alpha_2,p}g$ is the Hermite interpolating polynomial of degree $2p-1$ interpolating $g \in C^{(p-1)}[\alpha_1, \alpha_2]$ at α_1 and α_2.

By Theorem 3 of [8], Hf minimizes

$$\int_{\alpha_1}^{\alpha_2} \int_{\beta}^{d} [v^{(p,p)}(x, y)]^2 \, dx \, dy,$$

among all functions $v \in R_C^{p,p}$ satisfying

$$\frac{\partial^i v}{\partial n^i} = \frac{\partial^i f}{\partial n^i}, \qquad 0 \leqslant i \leqslant p-1$$

on $\partial\mathscr{U} \cap \partial(R\backslash\mathscr{U})$. Thus the left inverse F of the extension E given by (4.1) is norm reducing and thus E is a minimal extension for $\mathscr{U}$.

With the $\bar{x}_i$ chosen from $[a, \alpha_1] \cup [\alpha_2, b]$ and the $\bar{y}_j$ chosen from $[c, \beta]$; we have

$$[Ev, Ev] = \iint_{\mathscr{U}} [v^{(p,p)}(x, y)]^2 \, dx \, dy + \int_{\alpha_1}^{\alpha_2} \int_{\beta}^{d} [(Hv)^{(p,p)}(x, y)]^2 \, dx \, dy$$
$$+ \sum_{j=1}^{p} \int_{a}^{b} [v^{(p,0)}(x, \bar{y}_j)]^2 \, dx + \sum_{i=1}^{p} \int_{c}^{d} [v^{(0,p)}(\bar{x}_i, y)]^2 \, dy, \tag{4.2}$$

where

$$\int_{\alpha_1}^{\alpha_2} \int_{\beta}^{d} [(Hv)^{(p,p)}(x, y)]^2 \, dx \, dy = \int_{\beta}^{d} \Big\{ \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \gamma_{i,j,1} [v^{(i,p)}(\alpha_1, y) \, v^{(j,p)}(\alpha_1, y)]$$
$$+ 2 \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \gamma_{i,j,1,2} [v^{(i,p)}(\alpha_1, y) \, v^{(j,p)}(\alpha_2, y)]$$
$$+ \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \gamma_{i,j,2} [v^{(i,p)}(\alpha_2, y) \, v^{(j,p)}(\alpha_2, y)] \Big\} \, dy, \tag{4.3}$$

where

$$\gamma_{i,j,s} = \int_{\alpha_1}^{\alpha_2} h_{i,s}^{(p)}(x) \, h_{j,s}^{(p)}(x) \, dx, \qquad s = 1, 2,$$

and

$$\gamma_{i,j,1,2} = \int_{\alpha_1}^{\alpha_2} h_{i,1}^{(p)}(x) \, h_{j,2}^{(p)}(x) \, dx.$$

Here the $h_{i,s}$ are the cardinal functions for Hermite interpolation, i.e.,

$$H_{\alpha_1,\alpha_2,p}g(x) = \sum_{i=0}^{p-1} h_{i,1}(x)\, g^{(i)}(\alpha_1) + \sum_{i=0}^{p-1} h_{i,2}(x)\, g^{(i)}(\alpha_2),\ g \in C^{(p-1)}[\alpha_1, \alpha_2].$$

The interpolation conditions which determine $I^p_{\pi,\mathscr{U}}f$, are given by (2.5). For $\mathscr{U}$, the $\{\mu_i\}_1^m$ can be chosen to include the set of functionals

$$\mathscr{M}_0 = \{\lambda E \mid \lambda \in \Lambda \quad \text{and} \quad \lambda Q = \lambda\},$$

where $Q = EF$ and $Ff = f|_{\mathscr{U}}$.

The set $\mathscr{M}_0$ includes the functionals associated with (i) interpolation to values at each mesh point of $\mathscr{U}$; (ii) interpolation to the first $p - 1$ normal derivatives at each boundary mesh point of $\mathscr{U}$ which is also on the boundary of R; (iii) interpolation to cross-derivatives at the corners of R.

Because of our assumption that $\dim S^{2p}(\mathscr{U}, \pi) \leqslant S^{2p}(R, \pi)$, there must exist mesh points $\tilde{x}_1, \ldots, \tilde{x}_{2p-2}$ such that

$$\alpha_1 < \tilde{x}_1 < \cdots < \tilde{x}_{2p-2} < \alpha_2 .$$

The remaining interpolation conditions can be chosen to be the set of functionals

$$\mathscr{M}_1 = \{\lambda E \mid \lambda \in \Lambda^1\},$$

where

$$\Lambda^1 = \{\delta_{\tilde{x}_i} \otimes \delta_{y_j} \mid (\tilde{x}_i, y_j) \in \pi,\ i = 1, \ldots, 2p-2;\ \beta < y_j \leqslant d\}.$$
$$\bigcup \{\delta_{x_i} \otimes \delta_d{}^j,\ \alpha_1 < x_i < \alpha_2;\ j = 0, \ldots, p-1\}.$$

THEOREM 3. *Let $f \in C^{(2p,2p)}(\mathscr{U})$, $p = 2, 3$. Then there exists a constant K independent of π and f such that*

$$\|(f - I^p_{\pi,\mathscr{U}}f)^{(i,j)}\|_{\infty,\mathscr{U}} \leqslant K\bar{\pi}^{2p-l}\Big[\|f^{(2p,0)}\|_{\infty,\mathscr{U}} + \|f^{(0,2p)}\|_{\infty,\mathscr{U}} + \|f^{(2p,p)}\|_{\infty,\mathscr{U}} + \sum_{i=1}^{p} \|f^{(i,2p)}\|_{\infty,\mathscr{U}} + \|f^{(2p,2p)}\|_{\infty,\mathscr{U}}\Big],$$
$$l = \max\{i, j\}, \quad 0 \leqslant i,\ \ j \leqslant p-1. \tag{4.4}$$

Remark. Theorem 3 can be proved in much the same way as Theorem 2 except that for U-shaped regions $\|(Er)^{(p,p)}\|_{L^2(R)}$ does not simply reduce to $\|r^{(p,p)}\|_{L^2(\mathscr{U})}$. To take care of this complication we have had to assume more smoothness in f.

Proof of Theorem 3. In the same way as for the L-shaped region, we bound the terms on the right side of (2.12) individually. For each of the terms in the two sums we again obtain the inequalities (3.3) and (3.4). The quantities $\| r^{(p,0)}(t, \bar{y}_n)\|_\infty$, $n = 1,\ldots, p$, and $\| r^{(0,p)}(\bar{x}_k, u)\|_\infty$, $k = 1,\ldots, p$, can be bounded in the same way as in the proof of Theorem 2. In place of (3.5) we obtain for the U-shaped region

$$\left\| \int_a^b \int_c^d \partial^{i+j}/\partial x^i\, \partial y^j (K_{p,p}(x, y, t, u))(Er)^{(p,p)}(t, u)\, dt\, du \right\|_{\infty,\mathscr{U}} \leqslant (p-1)\, M_8 \bar{\pi}^{p-l} \|(Er)^{(p,p)}\|_{L^2(R)}, \qquad l = \max\{i, j\}. \tag{4.5}$$

It remains to obtain a suitable bound for $\|(Er)^{(p,p)}\|_{L^2(R)}$.

The arguments used to prove Lemma 2 give the following:

LEMMA 3. *Let* $s_f = I^p_{\pi,\mathscr{U}} f$. *For all* $s \in S^{2p}(\mathscr{U}, \pi)$

$$\|(E(f - s_f))^{(p,p)}\|_{L^2(R)} \leqslant \|(E(f - s))^{(p,p)}\|_{L^2(R)} \tag{4.6}$$

with equality if and only if $s = s_f + \eta$ *where* $(E\eta)^{(p,p)} \equiv 0$ *on* R.

Thus we can bound $\|(Er)^{(p,p)}\|_{L^2(R)}$ by obtaining an appropriate bound on

$$\| f^{(p,p)} - \hat{s}^{(p,p)}\|_{L^2(\mathscr{U})} + \|(H(f - \hat{s}))^{(p,p)}\|_{L^2([\alpha_1,\alpha_2]\times[\beta,d])}, \tag{4.7}$$

where $\hat{s} = (F_{\pi_1} \otimes F_{\pi_2})f$, where $F_{\pi_1} g$ is the univariate quasi-interpolant of degree $2p - 1$ to g as defined in [5].

The results of [5] imply that for $g \in C^{(2p)}[a, b]$,

$$\| g^{(i)} - (F_{\pi_1} g)^{(i)}\|_\infty \leqslant C_i \bar{\pi}_x^{2p-i} \| g^{(2p)}\|_{\infty,[a,b]}, \qquad 0 \leqslant i \leqslant 2p - 1, \tag{4.8}$$

where the C_i are independent of g and for $i \leqslant p$ the C_i are also independent of π_1. The arguments used to establish this result actually establish that for each $x \in [a, b]$

$$| g^{(i)}(x) - (F_{\pi_1} g)^{(i)}(x)| \leqslant C_i \bar{\pi}_x^{2p-i} \| g^{(2p)}\|_{\infty,I_x}, \qquad 0 \leqslant i \leqslant 2p - 1, \tag{4.9}$$

where I_x is a uniformly small neighborhood of x. By an appropriate choice of certain parameters, namely the τ_j, in the formula for the quasi-interpolant, for all $x \in [a, b]$ the intervals I_x will be contained in $[a, b]$.

For $(x, y) \in \mathscr{U}$

$$\begin{aligned} |(f - \hat{s})^{(i,j)}(x, y)| \leqslant{}& |((1 - F_{\pi_1}) f)^{(i,j)}(x, y)| + |((1 - F_{\pi_2}) f)^{(i,j)}(x, y)| \\ &+ |([(1 - F_{\pi_1}) \otimes (1 - F_{\pi_2})] f)^{(i,j)}(x, y)|, \\ &\qquad 0 \leqslant i, \quad j \leqslant p. \end{aligned}$$

From (4.9), one obtains

$$|((1 - F_{\pi_1})f)^{(i,j)}(x, y)| \leqslant C_i \bar{\pi}_x^{2p-i} \| f^{(2p,j)}(x, y)\|_{\infty, I_x},$$

$$|((1 - F_{\pi_2})f)^{(i,j)}(x, y)| \leqslant C_j \bar{\pi}_y^{2p-j} \| f^{(i,2p)}(x, y)\|_{\infty, I_y},$$

and

$$|([(1 - F_{\pi_1}) \otimes (1 - F_{\pi_2})]f)^{(i,j)}(x, y)| \leqslant C_i C_j \bar{\pi}^{4p-i-j} \| f^{(2p,2p)} \|_{\infty, I_x \times I_y}.$$

Again, by appropriate choice of the parameters, τ_j, in the formula for the quasi-interpolant, one has that for all $(x, y) \in \mathscr{U}$, $I_x \times I_y \in \mathscr{U}$. Thus

$$\|(f - \hat{s})^{(i,j)}\|_{\infty, \mathscr{U}} \leqslant K' \bar{\pi}^{2p-l} [\| f^{(2p,j)} \|_{\infty, \mathscr{U}} + \| f^{(i,2p)} \|_{\infty, \mathscr{U}} + \| f^{(2p,2p)} \|_{\infty, \mathscr{U}}],$$
$$0 \leqslant i, \quad j \leqslant p, \quad l = \max\{i, j\}, \quad (4.10)$$

where K' is some constant independent of f and π.

In (4.7), one is required to bound $(f - \hat{s})^{(p,p)}(x, y)$ on $\mathscr{U}$ and $(f - \hat{s})^{(i,p)}(\alpha_j, y)$, $j = 1, 2$; $0 \leqslant i \leqslant p - 1$, on $[\beta, d]$. The application of (4.10) to (4.7) along with the previously established bounds on the single integral terms in (2.12) leads to the estimate (4.4) and proves the theorem.

Remark. The techniques used to establish Theorems 2 and 3 can be used to prove similar results for other simple rectangular polygons, for example T- or H-shaped regions or regions similar to Fig. 4 of [12]. In fact these results can easily be extended to include any rectangular polygon such that the intersection with $R \backslash \mathscr{R}$ of the lines $x = \bar{x}_i$, $i = 1,..., p$, $y = \bar{y}_j$, $j = 1,..., p$ in the norm for $R_C^{p,p}$ is null, and for which one can show that $(Ef)^{(p,p)}$ on $R \backslash \mathscr{R}$ is either zero or can be expressed in terms of derivatives of f on $\mathscr{R}$.

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